

ON THE RING OF DIFFERENTIAL OPERATORS OF CERTAIN REGULAR DOMAINS

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ABSTRACT. Let (A, \mathfrak{m}) be a complete equicharacteristic Noetherian domain of dimension $d + 1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a regular local ring. Let $\text{Sing}(A)$ the singular locus of A be defined by an ideal J in A . Note $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d . We show R contains naturally a field $\ell \cong k((X))$. Let \mathfrak{g} be the set of ℓ -linear derivations of R and let $D(R)$ be the subring of $\text{Hom}_\ell(R, R)$ generated by \mathfrak{g} and the multiplication operators defined by elements in the ring R . We show that $D(R)$, the ring of ℓ -linear differential operators on R , is a left, right Noetherian ring of global dimension d . This enables us to prove Lyubeznik's conjecture on R modulo a conjecture on roots of Bernstein-Sato polynomials over power series rings.

1. INTRODUCTION

Let K be a field of characteristic zero and let R be a commutative Noetherian domain containing K as a subring. Let \mathfrak{g} be the set of K -linear derivations of R and let $D(R)$ be the subring of $\text{Hom}_K(R, R)$ generated by \mathfrak{g} and the multiplication operators defined by elements in the ring R . The ring $D(R)$ is called the ring of K -linear differential operators on R . In general $D(R)$ does not have good properties. However in the following cases it is known that $D(R)$ is both left and right Noetherian with finite global dimension:

- (1) $R = K[X_1, \dots, X_n]$. In this case $D(R) = A_n(K)$ the n^{th} -Weyl algebra over K . We have global dimension of $D(R)$ is equal to n , see [3, Chapter 2, Theorem 3.15].
- (2) $R = K[[X_1, \dots, X_n]]$. In this case global dimension of $D(R)$ is equal to n , see [3, Chapter 3, Proposition 1.8].
- (3) Let $K = \mathbb{C}$ and let V be a non-singular affine K -variety. Let R be the co-ordinate ring of V . In this case global dimension of $D(R)$ is equal to $\dim V$, see [3, Chapter 3, Theorem 2.5].
- (4) Let $R = \mathbb{C}\{z_1, \dots, z_n\}$ be the local ring of convergent power series in n -variables. In this case global dimension of $D(R)$ is equal to n , see [3, p. 197].

In this paper we describe a *new* vast class of Noetherian domains R with $D(R)$ both left and right Noetherian and with finite global dimension.

1.1. Setup: Let (A, \mathfrak{m}) be a complete equicharacteristic Noetherian domain of dimension $d + 1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a

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regular local ring. Let $\text{Sing}(A)$ the singular locus of A be defined by an ideal J in A . Note $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d . In 2.2 we show that R contains naturally a field $\ell \cong k((X))$. Let \mathfrak{g} be the set of ℓ -linear derivations of R and let $D(R)$ be the subring of $\text{Hom}_\ell(R, R)$ generated by \mathfrak{g} and the multiplication operators defined by elements in the ring R . The main result of this paper is

Theorem 1.2. *[with hypotheses as in 1.1] The ring $D(R)$ is a left, right Noetherian ring of global dimension d .*

1.3. Application: Lyubeznik conjectured, see [10], that if S is a regular ring and I is an ideal in S then for any $i \geq 0$ the set $\text{Ass}_S H_I^i(S)$ is finite. This conjecture is known to be true in the following cases:

- (1) S contains a field K with $\text{char } K = p > 0$, see [7].
- (2) S is local and containing a field K with $\text{char } K = 0$, see [9].
- (3) S is a regular affine K -algebra (here $\text{char } K = 0$), see [9].
- (4) S is an unramified regular local ring, see [11].
- (5) S is a smooth \mathbb{Z} -algebra, see [2].

However none of the techniques used to prove the above results can be used to verify Lyubeznik's conjecture for rings R as given in 1.1. We show that Theorem 1.2 and an affirmative answer to a question regarding Bernstein-Sato polynomials of a formal power series (see 5.7 and 5.8) enables us to verify Lyubeznik's conjecture in this case.

Here is an overview of the contents of the paper. In section two we discuss some preliminaries that we need. In section three we discuss our result on ranks of certain modules of derivations. The main result in this section is Theorem 3.3. We prove Theorem 1.2 in section four. In the next section we give an application of our result to Lyubeznik's conjecture. Finally in section we give bountiful number of examples of regular rings satisfying our hypothesis 1.1.

2. SOME PRELIMINARIES

Let (A, \mathfrak{m}) be a complete equicharacteristic Noetherian domain of dimension $d + 1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a regular local ring. Let $\text{Sing}(A)$ the singular locus of A be defined by an ideal J in A . Note $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d . In this section we prove some preliminary facts about R and A . We note that A contains a field isomorphic to k . For convenience we also denote it with k .

We first prove

Proposition 2.1. *[with hypotheses as above.] Let \mathfrak{n} be a maximal ideal in R . Then $\mathfrak{n} = \mathfrak{q}R$ where \mathfrak{q} is a prime ideal of height d in A . In particular $\dim R = d$.*

Proof. Note $f \notin \mathfrak{q}$. Suppose if possible $\text{height } \mathfrak{q} \leq d - 1$. As A is complete it is catenary. So $\text{height}(\mathfrak{m}/\mathfrak{q}) \geq 2$. In particular $\dim A/\mathfrak{q} \geq 2$. The image of f is non-zero in A/\mathfrak{q} . It is elementary to see that A/\mathfrak{q} has infinitely many prime ideals of height one. We can choose one, say $\overline{P} = P/\mathfrak{q}$ not containing \overline{f} . Thus P is a prime ideal in A not containing f and P strictly contains \mathfrak{q} . It follows that $\mathfrak{n} = \mathfrak{q}R$ is not a maximal ideal of R , a contradiction. \square

2.2. Consider the map $\phi: k[[X]] \rightarrow A$ which maps k identically to k and X to f . As A is a domain it is clear that ϕ is an injective map. Inverting X we get a map $\psi: k((X)) \rightarrow A_X$. It is clear that $A_X = A_f = R$. Thus R naturally contains a field $\ell \cong k((X))$. We also note that $\text{image } \phi = k[[f]]$ and $\ell = k((f))$.

The following is a crucial ingredient to prove Theorem 1.2.

Lemma 2.3. *Let \mathfrak{n} be a maximal ideal of R . Then R/\mathfrak{n} is a finite extension of ℓ .*

Proof. By Proposition 2.1 we get that $\mathfrak{n} = \mathfrak{q}R$ where \mathfrak{q} is a prime ideal of height d in A not containing f . The map $\phi: k[[X]] \rightarrow A$ as in 2.2 descends to a map $\bar{\phi}: k[[X]] \rightarrow A/\mathfrak{q}$. Set $T = A/\mathfrak{q}$ and $S = k[[X]]$. As (\mathfrak{q}, f) is \mathfrak{m} -primary in A we get that $T/XT = A/(\mathfrak{q}, f)$ is a finite dimensional k -vector space. We also get

$$\bigcap_{n \geq 1} X^n T \subseteq \bigcap_{n \geq 1} f^n T \subseteq \bigcap_{n \geq 1} \mathfrak{m}^n T = 0.$$

Thus T is separated with respect to (X) -topology of S . It follows that T is a finite S -module, see [12, Theorem 8.4]. Therefore the quotient field of A/\mathfrak{q} will be a finite extension of quotient field of S . The result follows. \square

We will use the next result in the next section.

Lemma 2.4. *[with hypotheses as above:] Let \mathfrak{q} be a prime of height d in A such that $f \notin \mathfrak{q}$. Let $\kappa(\mathfrak{q})$ be the residue field of $A_{\mathfrak{q}}$. Then there exists $y_1, \dots, y_d \in \mathfrak{q}$ such that*

- (1) $\text{height}(f, y_1, \dots, y_j) = j + 1$ for $j = 0, \dots, d$.
- (2) *The images of y_1, \dots, y_j in the $\kappa(\mathfrak{q})$ -vector space $\mathfrak{q}A_{\mathfrak{q}}/\mathfrak{q}^2A_{\mathfrak{q}}$ is linearly independent for $j = 1, \dots, d$.*
- (3) f, y_1, \dots, y_d is a system of parameters of A .
- (4) $(y_1, \dots, y_d)A_{\mathfrak{q}} = \mathfrak{q}A_{\mathfrak{q}}$.

Proof. (1) and (2). As A is a domain we get that $\text{height}(f) = 1$. Now suppose y_1, \dots, y_j is already chosen where $0 \leq j < d$. We choose y_{j+1} as follows:

(a) Let P_1, \dots, P_s be all the minimal primes of (f, y_1, \dots, y_j) of height $j + 1$. We claim that $\mathfrak{q} \not\subseteq P_i$ for all $i = 1, \dots, s$. We have to consider two cases:

case (i) : $j \leq d - 2$. Then as $\text{height } P_i < d$ for all i , we get the result.

case(ii): $j = d - 1$. If $\mathfrak{q} \subseteq P_i$ for some i then as both these prime ideals have height d we get $\mathfrak{q} = P_i$. We then get $f \in \mathfrak{q}$, a contradiction.

(b) Set

$$J = ((y_1, \dots, y_j)A_{\mathfrak{q}} + \mathfrak{q}^2A_{\mathfrak{q}}) \cap A.$$

Then $J \subseteq \mathfrak{q}$. We claim that $\mathfrak{q} \not\subseteq J$. If this is so we get $\mathfrak{q} = J$ and therefore

$$\mathfrak{q}A_{\mathfrak{q}} = (y_1, \dots, y_j)A_{\mathfrak{q}} + \mathfrak{q}^2A_{\mathfrak{q}} \quad \text{and so by Nakayama's Lemma } \mathfrak{q}A_{\mathfrak{q}} = (y_1, \dots, y_j)A_{\mathfrak{q}}.$$

This implies that $\dim A_{\mathfrak{q}} \leq j < d$, a contradiction.

By prime avoidance there exists

$$y_{j+1} \in \mathfrak{q} \setminus J \cup (\cup_{i=1}^s P_i).$$

Then note that y_1, \dots, y_{j+1} satisfies the conditions of (1) and (2).

(3) This follows since by (1) we have $\text{height}(f, y_1, \dots, y_d) = d + 1 = \dim A$.

(4) As $A_{\mathfrak{q}}$ is a regular local ring of dimension d we get that $\mathfrak{q}A_{\mathfrak{q}}/\mathfrak{q}^2A_{\mathfrak{q}}$ is a d -dimensional $\kappa(\mathfrak{q})$ -vector space. The result follows from (2). \square

3. RANKS OF MODULES OF DERIVATIONS

Let T, S be commutative Noetherian rings. Assume S is a T -algebra. Let $\text{Der}_T(S)$ denote the set of T -linear derivations on S . The S -module $\text{Der}_T(S)$ need not be finitely generated. However there are many natural instances where it is so.

3.1. Our setup in this section will be as in 1.1. Let us recall it here. Let (A, \mathfrak{m}) be a complete equicharacteristic Noetherian domain of dimension $d + 1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a regular local ring. Let $\text{Sing}(A)$ the singular locus of A be defined by an ideal J in A . Note $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d . We note that A contains a field isomorphic to k . For convenience we also denote it with k . By 2.2 R contains a field ℓ isomorphic to $k((X))$. Furthermore by 2.3 if \mathfrak{n} is a maximal ideal of R then the field R/\mathfrak{n} is a finite extension of ℓ . Also if $\mathfrak{n} = \mathfrak{q}R$ is a maximal ideal of R with \mathfrak{q} a prime ideal in A then $\text{height } \mathfrak{q} = d$, see 2.1.

We first prove:

Proposition 3.2. *[with hypotheses as in 3.1.] The A -module $\text{Der}_k(A)$ is finitely generated with rank $= d + 1$.*

Proof. By [12, Theorem 30.7], $\text{Der}_k(A)$ is a finitely generated A -module of rank $\leq d + 1$. Let $A = Q/\mathfrak{q}$ where $Q = k[[x_1, \dots, x_n]]$ and $\mathfrak{q} \subseteq (x_1, \dots, x_n)^2$ is a prime ideal in Q . Let $r = \text{height } \mathfrak{q}$. Then $n = d + 1 + r$.

Let T be a finitely generated A -module. By equation (6) in Theorem 25.2 of [12] we get an exact sequence of A -modules

$$0 \rightarrow \text{Der}_k(A, T) \rightarrow \text{Der}_k(Q, T) \rightarrow \text{Hom}_A(\mathfrak{q}/\mathfrak{q}^2, T).$$

We note that $\text{Der}_k(Q, T) \cong T^n$. Set $T = A$ in the above equation

$$0 \rightarrow \text{Der}_k(A) \rightarrow A^n \rightarrow \text{Hom}_A(\mathfrak{q}/\mathfrak{q}^2, A).$$

We localize the above equation at (0) . We note that $(\mathfrak{q}/\mathfrak{q}^2)_{(0)} \cong \mathfrak{q}Q_{\mathfrak{q}}/\mathfrak{q}^2Q_{\mathfrak{q}} \cong \kappa(\mathfrak{q})^r$, here $\kappa(\mathfrak{q})$ is the residue field of $Q_{\mathfrak{q}}$ (this is so as $Q_{\mathfrak{q}}$ is a regular local ring of dimension r). Note $\kappa(\mathfrak{q})$ is also the quotient field of A . So we have an exact sequence

$$0 \rightarrow \text{Der}_k(A)_{(0)} \rightarrow \kappa(\mathfrak{q})^n \rightarrow \kappa(\mathfrak{q})^r.$$

Therefore $\text{rank } \text{Der}_k(A) \geq n - r = d + 1$. The result follows. \square

The following is the main result of this section:

Theorem 3.3. *(with hypotheses as in 3.1.) Let T be the subring $k[[f]]$ of A . Consider $\text{Der}_T(A)$. Then*

- (1) $\text{Der}_T(A)$ is a finitely generated A -module and $\text{rank } \text{Der}_T(A) \geq d$.
- (2) $\text{Der}_T(A)_f = \text{Der}_{\ell}(R)$. In particular $\text{Der}_{\ell}(R)$ is finitely generated as a R -module.
- (3) Let \mathfrak{n} be a maximal ideal of R . Then
 - (a) $\text{Der}_{\ell}(R_{\mathfrak{n}}) = (\text{Der}_{\ell}(R))_{\mathfrak{n}}$.
 - (b) $\text{Der}_{\ell}(R_{\mathfrak{n}})$ is free $R_{\mathfrak{n}}$ -module of rank d .
- (4) $\text{Der}_{\ell}(R)$ is a projective R -module of rank d .

We will need the following two easily proved facts:

3.4. Fact 1: Let K be a field of characteristic zero and let $S = K[[X_1, \dots, X_n]]$. Let T be an S -module, not necessarily finitely generated, such that T is complete with respect to (X_1, \dots, X_n) -adic topology. Then $\text{Der}_K(S, T) \cong T^n$.

3.5. Fact 2: Let $R \subseteq S$ be an inclusion of Noetherian domains. Let I be an ideal in R such that R is complete with respect to I -adic topology. Let J be an ideal in S such that S is complete with respect to J -adic topology. Assume $IS \subseteq J$. Let $\{r_n\}$ be a convergent sequence in R (in the I -adic topology) with $r_n \rightarrow r$. Then $\{r_n\}$ considered as a sequence in S is convergent in the J -adic topology and $\{r_n\}$ converges to r in S .

We now give

Proof of Theorem 3.3. (1) Consider the inclusion of rings $k \subseteq T \subseteq A$. By equation (3) in Theorem 25.1 of [12], for any A -module W we have the following exact sequence of A -modules

$$(3.5.1) \quad 0 \rightarrow \text{Der}_T(A, W) \rightarrow \text{Der}_k(A, W) \rightarrow \text{Der}_k(T, W).$$

We now put $W = A$ in (3.5.1). Notice that $T \cong k[[X]]$. As A is complete with respect to \mathfrak{m} -adic topology it is also complete with respect to (f) -adic topology. So $\text{Der}_k(T, A) \cong A$, see 3.4. By Proposition 3.2 we get that $\text{Der}_k(A)$ is finitely generated as an A -module and $\text{rank Der}_k(A) = d + 1$. The result follows from (3.5.1).

We need some work to prove the remaining assertions:

Claim-1: $\text{Der}_T(A)_f \subseteq \text{Der}_\ell(R)$. In particular $\text{rank Der}_\ell(R) \geq d$. *Note we are not yet asserting that $\text{Der}_\ell(R)$ is finitely generated as a R -module.*

Remark: Let L be the quotient field of R . By the rank of a not-necessarily finitely generated R -module M , we mean the cardinality of a basis of the L -vector space $M \otimes_R L$.

It is elementary that $\text{Der}_T(A)_f \subseteq \text{Der}_T(R)$. Now $T = k[[f]]$ and f is invertible in R . Let $D \in \text{Der}_T(R)$. We assert that it is $\ell = k((f))$ -linear. To see this let

$$v = \frac{1}{f^i} r \quad \text{for some } i \geq 1 \text{ and } r \in R.$$

Then $r = f^i v$. As D is T -linear we get $D(r) = f^i D(v)$. It follows that

$$D(v) = \frac{1}{f^i} D(r).$$

Any $\xi \in \ell \setminus T$ is of the form t/f^i where $t \in T$ and $i \geq 1$. By the previous argument we get that $D(\xi r) = \xi D(r)$ for any $r \in R$. Thus D is ℓ -linear.

Now let \mathfrak{n} be a maximal ideal of R . Say $\mathfrak{n} = \mathfrak{q}R$ where \mathfrak{q} is a prime ideal in A of height d and $f \notin \mathfrak{q}$. By Lemma 2.4 there exists $y_1, \dots, y_d \in \mathfrak{q}$ such that f, y_1, \dots, y_d is a system of parameters of A and $(y_1, \dots, y_d)A_{\mathfrak{q}} = \mathfrak{q}A_{\mathfrak{q}}$. Set

$$V = k[[f, y_1, \dots, y_d]] = T[[y_1, \dots, y_d]]. \quad \text{Note } V \cong k[[Y_0, Y_1, \dots, Y_d]].$$

Also note that A is finitely generated as a V -module.

Claim-2: $\text{Der}_\ell(R_{\mathfrak{n}})$ is a free $R_{\mathfrak{n}}$ -module of rank d . There also exists $\delta_i \in \text{Der}_\ell(R_{\mathfrak{n}})$ such that

$$\delta_i(y_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

(In particular $\delta_1, \dots, \delta_d$ generate $\text{Der}_\ell(R_{\mathfrak{n}})$ as a $R_{\mathfrak{n}}$ -module).

We note that $(\text{Der}_\ell(R))_{\mathfrak{n}} \subseteq \text{Der}_\ell(R_{\mathfrak{n}})$. By Claim 1: we get $\text{rank Der}_\ell(R_{\mathfrak{n}}) \geq d$ as a $R_{\mathfrak{n}}$ -module. Using [12, Theorem 30.7] and Lemma 2.3 we get that $\text{rank Der}_\ell(R_{\mathfrak{n}}) \leq$

d as a $R_{\mathfrak{n}}$ -module. So $\text{rank Der}_{\ell}(R_{\mathfrak{n}}) = d$. Set $z_i = \text{image of } y_i \text{ in } A_{\mathfrak{q}}$. As $R_{\mathfrak{n}} = A_{\mathfrak{q}}$ is regular local and z_1, \dots, z_d is a regular system of parameters of $A_{\mathfrak{q}}$, by [12, Theorem 30.6] we get that There also exists $\delta_i \in \text{Der}_{\ell}(R_{\mathfrak{n}})$ such that

$$(3.5.2) \quad \delta_i(z_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

The result follows since as A is a domain we get $A \subseteq A_{\mathfrak{q}}$.

By an argument similar to that in Claim 1 we get $\text{Der}_T(A)_{\mathfrak{q}} \subseteq \text{Der}_{\ell}(A)_{\mathfrak{q}}$. More is true. In fact we

Claim-3: For $i = 1, \dots, d$ there exists $D_i \in \text{Der}_T(A)$ such that $\delta_i = D_i/s_i$ for some $s_i \notin \mathfrak{q}$. In particular $\text{Der}_T(A)_{\mathfrak{q}} = \text{Der}_{\ell}(A_{\mathfrak{q}})$.

We prove it for $i = 1$. The argument for $i > 1$ is similar. Set $W = T[y_1, \dots, y_d]$. As δ_1 is T -linear and $\delta_1(y_j) = 1$ if $j = 1$ and 0 if $j > 1$ we get that restricted map $(\delta_1)_W \in \text{Der}_T(W)$. We note that $W \cong T[Y_1, \dots, Y_d]$ and $(\delta_1)_W$ is usual differentiation with respect to Y_1 .

We now

Claim-4: $\delta_1(V) \subseteq V$.

Assume the claim for the moment. Now A is finitely generated as a V -module. Say $A = Va_1 + \dots + Va_c$. Say $\delta_1(a_j) = u_j/t_j$ where $a_j, t_j \in A$ and $t_j \notin \mathfrak{q}$. Set $D_1 = s_1\delta_1$ where $s_1 = t_1 \dots t_c$. Notice $D_1(V) \subseteq A$. Also $D_1 \in \text{Der}_{\ell}(A_{\mathfrak{q}})$. It is clear that $D_1(A) \subseteq A$ and D_1 is T -linear. Thus $(D_1)_A \in \text{Der}_T(A)$ and so $D_1 = s_1\delta_1$ in $\text{Der}_{\ell}(A_{\mathfrak{q}})$. The result follows.

We now give a proof of Claim 4:

Set $K = R/\mathfrak{n} = \kappa(\mathfrak{q})$ the residue field of $A_{\mathfrak{q}}$. Note that the $\mathfrak{q}A_{\mathfrak{q}}$ completion of $A_{\mathfrak{q}}$ is $\widehat{A}_{\mathfrak{q}} = K[[z_1, \dots, z_d]]$. (Recall that z_i is the image of y_i in $A_{\mathfrak{q}}$). Furthermore δ_1 extends to a K -linear derivation on $\widehat{A}_{\mathfrak{q}}$ and it is in fact differentiation with respect to z_1 .

We now note that we have an inclusion of rings $V = T[[y_1, \dots, y_d]] \subseteq \widehat{A}_{\mathfrak{q}}$. Furthermore V is complete with respect to $I = (y_1, \dots, y_d)$ and $I\widehat{A}_{\mathfrak{q}} = \mathfrak{q}\widehat{A}_{\mathfrak{q}}$. Let $\xi \in V$. Write

$$\xi = \sum_{j \geq 0} t_j y_1^j \quad \text{where } t_j \in T[[y_2, \dots, y_d]].$$

Set

$$\xi_n = \sum_{j=0}^n t_j y_1^j.$$

Notice $\xi_n \in W$ and $\xi_n \rightarrow \xi$ in V (with respect to the I -adic topology on V). Set

$$\eta = \sum_{j \geq 1} j t_j y_1^{j-1} \quad \text{and} \quad \eta_n = \sum_{j=1}^n j t_j y_1^{j-1}.$$

We note that $\eta_n \rightarrow \eta$ in V .

By 3.5 we get that $\xi_n \rightarrow \xi$ in $\widehat{A}_{\mathfrak{q}}$. As δ_1 is continuous with respect to $\mathfrak{q}\widehat{A}_{\mathfrak{q}}$ -adic topology in $\widehat{A}_{\mathfrak{q}}$ we get that $\delta_1(\xi_n) \rightarrow \delta_1(\xi)$. Notice $\delta_1(\xi_n) = \eta_n$. It follows (using 3.5) that $\delta_1(\xi) = \eta$. Thus $\delta_1(V) \subseteq V$ and we have proved Claim 4.

(2) We have an inclusion of R -modules $\text{Der}_T(A)_f \subseteq \text{Der}_{\ell}(R)$. If $\mathfrak{n} = \mathfrak{q}R$ is a maximal ideal in R (where \mathfrak{q} is a height d prime ideal in A and $f \notin \mathfrak{q}$) then

we have $\text{Der}_\ell(R)_\mathfrak{n} \subseteq \text{Der}_\ell(R_\mathfrak{n})$. Note $R_\mathfrak{n} = A_\mathfrak{q}$ and by Claim 3 we have that $\text{Der}_T(A)_\mathfrak{q} = \text{Der}_\ell(A_\mathfrak{q})$. In particular we have $(\text{Der}_T(A)_f)_\mathfrak{n} = (\text{Der}_\ell(R))_\mathfrak{n}$ for every maximal ideal \mathfrak{n} of R . Therefore $\text{Der}_T(A)_f = \text{Der}_\ell(R)$.

(3) (a). This follows from (2) and Claim 3.

(3)(b). This follows from Claim 2 and 3(a).

(4). This follows from (3). \square

4. PROOF OF THEOREM 1.2

In this section we prove our main Theorem. Let us first recall a result from the influential book [3].

4.1. Let K be a field of characteristic zero and let R be a commutative Noetherian domain containing K as a subring. Let \mathfrak{g} be the set of K -linear derivations of R and let $D(R)$ be the subring of $\text{Hom}_K(R, R)$ generated by \mathfrak{g} and the multiplication operators defined by elements in the ring R .

Let \mathfrak{m} be a maximal ideal of R . If $\delta \in \mathfrak{g}$ then $\delta(\mathfrak{m}^2) \subseteq \mathfrak{m}$ and so δ induces a R/\mathfrak{m} -linear map from $\mathfrak{m}/\mathfrak{m}^2$ to R/\mathfrak{m} which is called the *tangent map* of δ at \mathfrak{m} . We say that \mathfrak{g} has *maximal rank* at \mathfrak{m} if every R/\mathfrak{m} linear map from $\mathfrak{m}/\mathfrak{m}^2$ to R/\mathfrak{m} is the tangent map of some $\delta \in \mathfrak{g}$.

Now consider the following conditions:

- (1) \mathfrak{g} has maximal rank at every maximal ideal in R .
- (2) There exists an integer n such that $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq n$ for all maximal ideals \mathfrak{m} of R and equality holds for some \mathfrak{m} .
- (3) The residue fields R/\mathfrak{m} are algebraic over K for all maximal ideals \mathfrak{m} of R .
- (4) If M is a R -module and if $M_\mathfrak{m} = M \otimes_R R_\mathfrak{m}$ is finitely generated as a $R_\mathfrak{m}$ -module for all maximal ideals \mathfrak{m} of R then M is finitely generated R -module.

Then the following is [3, Chapter 2, Theorem 1.2]:

Theorem 4.2. [with hypotheses as in 4.1] *If the conditions (1)-(4) hold then $D(R)$ is left and right Noetherian and global dimension of $D(R)$ is n .*

Remark 4.3. Only condition (4) above is difficult to verify. However it is used only to prove \mathfrak{g} is finitely generated as a R -module. Thus if we can independently verify that \mathfrak{g} is a finitely generated R -module then in Theorem 4.2 all we need is conditions (1)-(3).

We now give

Proof of Theorem 1.2. By Theorem 3.3 we get that $\mathfrak{g} = \text{Der}_\ell(R)$ is finitely generated as a R -module. Thus by Remark 4.3 we only need to verify conditions (1)-(3) above.

By Proposition 2.1 we get that $\dim_{R/\mathfrak{n}} \mathfrak{n}/\mathfrak{n}^2 = d$ for each maximal ideal of \mathfrak{n} of R . Also by Lemma 2.3 we get that R/\mathfrak{n} is a finite extension of ℓ for each maximal ideal \mathfrak{n} of R . Thus we have verified conditions (2) and (3).

Let \mathfrak{n} be a maximal ideal of R . Then by Theorem 3.3 we get that $\text{Der}_\ell(R)_\mathfrak{n} = \text{Der}_\ell(R_\mathfrak{n})$. Again by Theorem 3.3 we get that $\text{Der}_\ell(R_\mathfrak{n})$ is a free $R_\mathfrak{n}$ -module of rank d . Furthermore by (3.5.2) there exists a regular system of parameters z_1, \dots, z_d of

$R_{\mathfrak{n}}$ and $\delta_1, \dots, \delta_d \in \text{Der}_{\ell}(R_{\mathfrak{n}})$ such that

$$(4.3.3) \quad \delta_i(z_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

We note that $z_1 + \mathfrak{n}^2, \dots, z_d + \mathfrak{n}^2$ is a basis of the R/\mathfrak{n} -vector space $\mathfrak{n}/\mathfrak{n}^2$.

As $\text{Der}_{\ell}(R)_{\mathfrak{n}} = \text{Der}_{\ell}(R_{\mathfrak{n}})$ there exists $D_i \in \text{Der}_{\ell}(R)$ and $s_i \notin \mathfrak{m}$ such that $\delta_i = D_i/s_i$ for $i = 1, \dots, d$. Let $\overline{s_i}$ denote the image of s_i in R/\mathfrak{n} . Note $\overline{s_i} \neq 0$ for all i . Thus by (4.3.3) we get

$$(4.3.4) \quad D_i(z_j) = \begin{cases} \overline{s_i} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

It is now elementary to show that \mathfrak{g} has maximal rank at \mathfrak{n} . Thus we have verified (1). The result now follows from Theorem 4.2. \square

We now ask:

Question 4.4. (with hypotheses as in Theorem 1.2). Is $D(R)$ a domain?

5. LYUBEZNIK'S CONJECTURE FOR RINGS CONSIDERED IN THIS PAPER

Let R be the regular domain as in 1.1 and let $D(R)$ be the ring of ℓ -linear differential operators on R . We first show that if Question 5.2 has an affirmative answer then Lyubeznik's conjecture holds for R . We then show that a positive answer to a question on Bernstein Sato polynomials of power series will enable us to solve Question 5.2.

5.1. We note that R can be considered both as a subring of $D(R)$ and also as a $D(R)$ -module. Furthermore it is clear that R is finitely generated as a $D(R)$ -module. Let $h \in R$ with $h \neq 0$. The usual arguments yield that R_h is a $D(R)$ -module. We ask

Question 5.2. Is R_h finitely generated as a $D(R)$ -module?

We now show:

Theorem 5.3. If Question 5.2 has an affirmative answer then Lyubeznik's conjecture holds for R , i.e., if I is any ideal in R then $\text{Ass}_R H_I^i(R)$ is finite for any ideal I in R and for $i \geq 0$.

For this we need the following two results:

Lemma 5.4. Let I be an ideal in R . Then for each $i \geq 0$ the local cohomology module $H_I^i(R)$ is a $D(R)$ -module. Furthermore if Question 5.2 has an affirmative answer then $H_I^i(R)$ is finitely generated as a $D(R)$ -module for all $i \geq 0$.

Proof. Let $I = (h_1, \dots, h_s)$ and consider the modified Čech complex \mathbf{C} on h_1, \dots, h_s . Note that \mathbf{C}_i is a finite direct sum of modules $R_{h_{j_1} \dots h_{j_i}}$ and the maps are the natural ones upto a sign. It follows that \mathbf{C} is in fact a complex of (left)- $D(R)$ -modules. It follows that $H_I^i(R) = H^i(\mathbf{C})$ is a $D(R)$ -module.

If Question 5.2 has an affirmative answer then \mathbf{C}_i is finitely generated $D(R)$ -module for each $i \geq 0$. As $D(R)$ is left Noetherian it follows that $H^i(\mathbf{C}) = H_I^i(R)$ is finitely generated as a $D(R)$ -module. \square

We also need the following result:

Lemma 5.5. *Let M be a finitely generated $D(R)$ -module. Then $\text{Ass}_R M$ is a finite set.*

Proof. Let I be an ideal in R . Set

$$\Gamma_I(M) = \{m \in M \mid I^n m = 0, \text{ for some } n \geq 1\} = H_I^0(M).$$

Then $\Gamma_I(M)$ is a $D(R)$ -submodule of M .

The following is a standard argument for proving finiteness of associate primes of modules which are Noetherian over a ring of Differential operators, for instance see [9]. We give it here for the convenience of the reader.

We claim there is a finite filtration of M by $D(R)$ -submodules $0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M_{s-1} \subseteq M_s = M$ such that M_j/M_{j-1} has only one associated prime for $j = 1, \dots, s$. For let P_1 be a maximal element in the set of the associated primes of M . Then $\Gamma_{P_1}(M)$ is non-zero and has only one associated prime, namely, P_1 . Set $M_1 = \Gamma_{P_1}(M)$. As argued before M_1 is a $D(R)$ -submodule of M , so M/M_1 is a $D(R)$ -module. Let P_2 be a maximal element in the set of the associated primes of M/M_1 . Then $\Gamma_{P_2}(M/M_1)$ is a non-zero $D(R)$ -submodule of M/M_1 and has only one associated prime, namely, P_2 . Set M_2 to be the preimage of $\Gamma_{P_2}(M/M_1)$ in M . Since M is Noetherian this process eventually stops. This proves the claim. The set of the associated primes of M is contained in the union of the sets of the associated primes of all M_i/M_{i-1} where $i = 1, \dots, s$. This proves our Lemma. \square

We now give:

Proof of Theorem 5.3. This follows from Lemma's 5.4 and 5.5. \square

5.6. *A relation between Question 5.2 and Bernstein-Sato polynomial.* Let K be a field of characteristic zero and let $S = K[[X_1, \dots, X_n]]$. Let $D(S)$ be the ring of K -linear differential operators over S . Let $h \in S$ be a non-unit and let $b_h(z)$ be it's Bernstein-Sato polynomial (see [3, Chapter 3, Corollary 3.6] and also [3, Chapter 1, Remark 5.8]). Let $-c \in \mathbb{Z}$ is a lower bound negative integer root of $b_h(z)$. Then it is easy to verify that S_h is generated as a $D(S)$ -module by h^{-c} , see [1, p. 460]. If $h \in S$ is in fact a polynomial then $-n$ is a lower bound for the roots of $b_h(z)$, see [16]. Set $bs(h)$ to be the smallest negative integer root of $b_h(z)$.

Set $\mathfrak{m} = (X_1, \dots, X_n)$ and if $h \in S$ is non-zero then set

$$v(h) = \max\{r \mid h \in \mathfrak{m}^r\}.$$

This is a non-negative integer, since by Krull's intersection theorem we have $\bigcap_{r \geq 1} \mathfrak{m}^r = 0$. We now state our next:

Question 5.7. *[with hypotheses as in 5.6.] Let m be a positive integer. Is*

$$K(m) = \sup\{bs(h) \mid v(h) \leq m, \text{ where } h \in S\} \text{ finite?}$$

I believe that to answer this question it suffices to consider the case $K = \mathbb{C}$, the complex numbers. Motivated by this we make our final

Question 5.8. *[with hypotheses as in Question 5.7.] Does there exists $c > 0$ such that $K(m) \leq c$ for any field K of characteristic zero?*

We now state the main result of this section

Theorem 5.9. *If Question 5.8 has an affirmative answer for all positive integers m then so does Question 5.2.*

Proof. Let \mathfrak{n} be a maximal ideal of R . As R is a domain we have $\cap_{i \geq 1} \mathfrak{n}^i = 0$, see [12, 8.10(ii)]. Let $h \in R$ be non-zero. Then

$$v_{\mathfrak{n}}(h) = \max\{i \mid h \in \mathfrak{n}^i\} \text{ is a non-negative integer.}$$

We now *Claim-1*: The set $V(h) = \{v_{\mathfrak{n}}(h) \mid \mathfrak{n} \text{ a maximal ideal of } R\}$ is bounded above.

Proof of Claim-1: Suppose if possible Claim-1 is not true. Then for any positive integer i there exists a maximal ideal \mathfrak{n}_i with $v_{\mathfrak{n}_i}(h) \geq i$.

As R is the localization of a complete local ring it is excellent. So by [6] there exists $c > 0$ such that

$$\mathfrak{n}^j \cap (h) = \mathfrak{n}^{j-c}((\mathfrak{n}^c \cap (h))) \text{ for all } j \geq c \text{ and for all maximal ideals } \mathfrak{n} \text{ of } R.$$

Choose \mathfrak{n}_i with $v_{\mathfrak{n}_i}(h) > c + 2$. Then we have

$$(h) = \mathfrak{n}_i^{c+1} \cap (h) = \mathfrak{n}_i(\mathfrak{n}_i^c \cap (h)) = \mathfrak{n}_i(h).$$

It follows that there exists $\xi \in \mathfrak{n}_i$ with $(1 - \xi)(h) = 0$. As R is a domain and $h \neq 0$ this forces $1 - \xi = 0$ and so $\xi = 1 \in \mathfrak{n}_i$. This is a contradiction. Thus Claim-1 is true.

Let m be an upper bound for $V(h)$. As we are assuming that Question 5.8 has an affirmative answer we get that there exists $c > 0$ such that $K(m) \leq c$ for any field K of characteristic zero.

Consider the the following ascending chain \mathbb{F} of $D(R)$ -submodules of R_h whose union is R_h

$$D(R)\frac{1}{h} \subseteq D(R)\frac{1}{h^2} \subseteq \cdots \subseteq D(R)\frac{1}{h^c} \subseteq \cdots \subseteq D(R)\frac{1}{h^p} \subseteq \cdots$$

We say \mathbb{F} stabilizes at level q if $D(R)h^{-p} = D(R)h^{-q}$ for all $p \geq q$.

Let \mathfrak{n} be a maximal ideal of R . We localize \mathbb{F} at \mathfrak{n} to get the ascending chain $\mathbb{F}_{\mathfrak{n}}$. We then tensor it with $\widehat{R}_{\mathfrak{n}}$, the completion of $R_{\mathfrak{n}}$, to get the ascending chain $\widehat{\mathbb{F}}_{\mathfrak{n}}$. Set $\kappa(\mathfrak{n}) = R/\mathfrak{n}$. Note $\widehat{R}_{\mathfrak{n}} \cong \kappa(\mathfrak{n})[[Z_1, \dots, Z_d]]$. Let $D(\widehat{R}_{\mathfrak{n}})$ be the ring of $\kappa(\mathfrak{n})$ -linear differential operators on $\widehat{R}_{\mathfrak{n}}$. Then by [3, Chapter 3, Lemma 1.5] we get that

$$D(R) \otimes_R \widehat{R}_{\mathfrak{n}} \cong D(\widehat{R}_{\mathfrak{n}})$$

and we also get that $\widehat{\mathbb{F}}_{\mathfrak{n}}$ is an ascending chain of $D(\widehat{R}_{\mathfrak{n}})$ -submodules of $(\widehat{R}_{\mathfrak{n}})_h$. It follows from 5.6 that $\widehat{\mathbb{F}}_{\mathfrak{n}}$ stabilizes at level c . As the map $R_{\mathfrak{n}} \rightarrow \widehat{R}_{\mathfrak{n}}$ is faithfully flat we get that $\mathbb{F}_{\mathfrak{n}}$ stabilizes at level c .

We have shown that $\mathbb{F}_{\mathfrak{n}}$ stabilizes at level c for any maximal ideal \mathfrak{n} of R . It follows that \mathbb{F} stabilizes at level c . Therefore R_h is generated as a $D(R)$ -module by $1/h^c$. In particular R_h is finitely generated as a $D(R)$ -module. \square

6. EXAMPLES

In this section we show that for each $d \geq 1$ there exist infinitely many examples of regular rings which satisfy our hypothesis 1.1. For simplicity we will assume that k is an algebraically closed field of characteristic zero.

Example 6.1. Let $Q = k[x_1, \dots, x_d, x_{d+1}]$ where $d \geq 2$. Set

$S = \widehat{Q} = k[[x_1, \dots, x_d, x_{d+1}]]$. Let $n \geq 2$ be a positive integer and let ζ be a primitive n^{th} -root of unity and let $G = \langle \zeta^i : 0 \leq i \leq n-1 \rangle$. Then G acts on both Q and S with the action $x_i \mapsto \zeta x_i$. Let $B = Q^G$ and $A = S^G$. Note that $B \cong Q^{<n>}$ the n^{th} Veronese subring of Q and that $A = \widehat{B}$ the completion of B at

it's irrelevant maximal ideal. As $\text{Proj}(B)$ is smooth we get that A is an isolated singularity. It is well known that $Cl(B)$, the class group of B is $\mathbb{Z}/n\mathbb{Z}$. As $\text{Proj}(B)$ is smooth and $\dim B = d + 1 \geq 3$ we get that B satisfies R_2 property of Serre. So by a result of Flenner, $Cl(A) \cong Cl(B)$.

Let $f = x_1^n + \cdots + x_{d+1}^n$. As $d \geq 2$, it is well-known that f is irreducible in Q . Note $f \in A$. Let \mathfrak{m} be the maximal ideal of S . If T is a quotient ring of S then set $G(T) = \bigoplus_{n \geq 0} \mathfrak{m}^n T / \mathfrak{m}^{n+1} T$ the associated graded ring of T with respect to its maximal ideal $\mathfrak{m}T$. Note $G(S/fS) \cong G(S)/fG(S) = Q/fQ$ which is a domain. So S/fS is a domain. In particular fS is a prime ideal in S . As $fA = fS \cap A$ we get that fA is a prime ideal in A .

Set $R_{n,d} = A_f$. By the localization sequence of class groups we have $Cl(R_{n,d}) = \mathbb{Z}/n\mathbb{Z}$. Also note that $\dim R_{n,d} = d \geq 2$.

In 6.1 we had the restriction that $d \geq 2$ and that R is not a UFD. Next we give infinitely many one dimensional examples satisfying 1.1. We also give infinitely many examples satisfying 1.1 of dimension $d \geq 3$ which are also UFD's. We need to recall the notion of simple singularities.

6.2. Simple singularities: Let $S = k[[x, y, z_2, \dots, z_d]]$ with $d \geq 2$. Simple singularities are defined by the following equations:

$$(A_n) \quad x^2 + y^{n+1} + \sum_{j=2}^d z_j^2 \quad (n \geq 1),$$

$$(D_n) \quad x^2 y + y^{n-1} + \sum_{j=2}^d z_j^2 \quad (n \geq 4),$$

$$(E_6) \quad x^3 + y^4 + \sum_{j=2}^d z_j^2,$$

$$(E_7) \quad x^3 + xy^3 + \sum_{j=2}^d z_j^2,$$

$$(E_8) \quad x^3 + y^5 + \sum_{j=2}^d z_j^2.$$

6.3. Let $A = Q/(f)$ be a simple singularity. Then A is an isolated singularity. In particular by a result due to Grothendieck A is a UFD if $\dim A \geq 4$. We also note that if $d \geq 2$ then $A/(z_d)$ is a simple singularity of the same type.

6.4. Grothendieck Groups: Let T be a commutative Noetherian ring and let $\text{mod}(T)$ denote the category of all finitely generated T -modules. Let \mathfrak{U} be an additive subcategory of $\text{mod}(T)$ closed under extensions and let $\text{Gr}(\mathfrak{U})$ denote the *Grothendieck group* of \mathfrak{U} . We recall the following three facts of Grothendieck groups that we need.

- (1) Let (A, \mathfrak{m}) be a Cohen-Macaulay local domain. Let \mathfrak{C} be the additive subcategory of $\text{mod}(A)$ consisting of all maximal Cohen-Macaulay A -modules. Then
 - (a) The inclusion $i : \mathfrak{C} \rightarrow \text{mod}(A)$ induces an isomorphism of Grothendieck groups $\text{Gr}(\mathfrak{C})$ and $\text{Gr}(\text{mod}(A))$, cf. [17, 13.2].

- (b) The map $\text{rk}: \text{Gr}(\mathfrak{C}) \rightarrow \mathbb{Z}$ defined by $[M] \mapsto \text{rank}(M)$ is well-defined surjective group homomorphism. We have an isomorphism $\mathbb{Z} \oplus \ker \text{rk} \rightarrow \text{Gr}(\mathfrak{C})$ where $(1, 0) \mapsto [A]$.
- (2) Let T be a regular ring of finite Krull dimension and let $K(T)$ be its K-group. Then the natural map $K(T) \rightarrow \text{Gr}(\text{mod}(T))$ is an isomorphism.
- (3) Let $f \in T$. The sequence

$$\text{Gr}(T/(f)) \xrightarrow{d_1} \text{Gr}(T) \xrightarrow{d_0} \text{Gr}(T_f) \rightarrow 0$$

is exact. Here

$$d_1([M]) = [M] \quad \text{and} \quad d_0([N]) = [N_f].$$

Remark 6.5. If f is T -regular then note that the class of $[T/(f)]$ is zero in $\text{Gr}(T)$. The reason is that we have an exact sequence $0 \rightarrow T \xrightarrow{f} T \rightarrow T/(f) \rightarrow 0$.

Remark 6.6. The Grothendieck groups of all simple singularities is known, see [17, 13.10]. We will only need the following fact: Let A be an A_n singularity of dimension l . Then

- (1) If n is even then $\text{Gr}(A) = \mathbb{Z}$ if l is odd and is equal to $\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$ if l is even.
- (2) If n is odd then $\text{Gr}(A) = \mathbb{Z}^2$ if l is odd and is equal to $\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$ if l is even.

Example 6.7. Let $S = k[[x, y, z_2, \dots, z_d]]$ with $d \geq 2$ and let $A = S/(f)$ be an A_n -singularity with n even. Note $\dim A = d + 1$. Set $R_{n,d} = A_{z_d}$. We note that if $\dim A \geq 4$ then A is a UFD and so $R_{n,d}$ is also a UFD.

Case 1: $\dim A = d + 1$ is even.

Consider the exact sequence

$$\text{Gr}(A/(z_d)) \xrightarrow{d_1} \text{Gr}(A) \xrightarrow{d_0} \text{Gr}(R_{n,d}) \rightarrow 0.$$

Note $A/(z_d)$ is an A_n singularity of dimension d . Also for all $d \geq 1$ the ring $A/(z_d)$ is a domain. By 6.4 and 6.6 we have that $\text{Gr}(A/(z_d)) = \mathbb{Z}$ and is generated by the class of $A/(z_d)$. By 6.4(3) it follows that $d_1 = 0$. It follows that

$$\mathbb{Z} \oplus \mathbb{Z}/(n+1) = \text{Gr}(A) = \text{Gr}(R_{n,d}) \cong K(R_{n,d}).$$

Case 2: $\dim A = d + 1$ is odd.

We again consider the exact sequence

$$\text{Gr}(A/(z_d)) \xrightarrow{d_1} \text{Gr}(A) \xrightarrow{d_0} \text{Gr}(R_{n,d}) \rightarrow 0.$$

We again assert that $d_1 = 0$. Notice $\text{Gr}(A/(z_d)) = \mathbb{Z} \oplus \mathbb{Z}/(n+1)$ and $\text{Gr}(A) = \mathbb{Z}$. Clearly $d_1(\mathbb{Z}/(n+1)) = 0$. By 6.4(1)(b) the element $(1, 0)$ of $\text{Gr}(A/(z_d))$ is generated by the class of $A/(z_d)$. By 6.4(3) we get $d_1([A/(z_d)]) = 0$. Thus again $d_1 = 0$. So we have

$$\mathbb{Z} = \text{Gr}(A) \cong \text{Gr}(R_{n,d}) \cong K(R_{n,d}).$$

Remark 6.8. The point of this section was to show that there exists infinitely many non-isomorphic regular rings satisfying our hypothesis 1.1. I believe that the examples given above is only a tip of the iceberg. There should be many more examples. However I do not know how to prove they are non-isomorphic.

Remark 6.9. The reason why the above remark is pertinent is due to a related result which we now describe. Let $S = \bigoplus_{n \geq 0} S_n$ be a standard graded algebra over an *uncountable* algebraically closed field $k = S_0$ with $\text{Proj}(S)$ smooth. Then there exists an uncountable family $\{f_\alpha \mid \alpha \in \Gamma\}$ of homogeneous elements of positive degree with $S_\alpha \not\cong S_\beta$ for $\alpha, \beta \in \Gamma$ and $f_\alpha \neq f_\beta$. This follows from a result in [8].

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